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Gauged gravity via spectral asymptotics of non-Laplace type operators

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Abstract: We construct invariant differential operators acting on sections of vector bundles of densities over a smooth manifold without using a riemannian metric. The spectral invariants of such operators are invariant under both the diffeomorphisms and the gauge transformations and can be used to induce a new theory of gravitation. It can be viewed as a matrix generalization of Einstein general relativity that reproduces the standard Einstein theory in the weak deformation limit. Relations with various mathematical constructions such as Finsler geometry and Hodge-de Rham theory are discussed.

Keywords: Classical Theories of Gravity, Models of Quantum Gravity, Differential and Algebraic Geometry, Non-Commutative Geometry.

Contents

1. Introduction

Einstein General Relativity is accepted as a correct theory of gravity at huge range of scales, from cosmological to the subatomic ones, that successfully describes almost all classical gravitational phenomena (with few exceptions like singularities, dark energy, etc.). In spite of this, there is no consistent theory of quantum gravitational phenomena, that is of physics at very small length scales. It is expected that the general relativistic description of gravity, and, as the result, of the space-time, is inadequate at short distances (and maybe at very large distances due to the cosmological constant problem). There are many different proposals how to extend General Relativity. Aside from the well established approaches such as string theory and loop gravity some work has been done in noncommutative theory of gravity associated with noncommutative spaces [1]–[4] as well as "matrix gravity" [5]–[13].

One of the first attempts of constructing matrix gravity was proposed by Okubo and Maluf [5], whose primary goal was to unify Einstein's general relativity and Yang-Mills gauge theory. The main ingredient in their approach was the matrix valued affine connection and the main purpose of their work was the unification of Einstein general relativity and Yang-Mills theory. They introduced the matrix-valued metric (or the frame vectors) and the connection coefficients and developed the formalism of some kind of "matrix-valued" differential geometry, that is, covariant derivatives, curvatures etc. Because of the non-commutativity many choices have to be made in any "matrix" differential geometry, such as how to raise indices, how to define covariant derivatives, what constraints to use to fix the torsion and some others. Different choices lead to different physical theories and there is no physical principle which could help in distinguishing a unique possibility. The problems encountered by Okubo and Maluf were: the presence of the spin-3 components of the connection, non-uniqueness of connection, non-covariance of the torsion, non-covariance of higher covariant derivatives, consistency of covariant derivatives of higher rank tensors and some others. The further work in this direction [6] was focused on an particular ansatz for the affine connection which was introduced originally by Einstein and Kaufman [14] in the abelian theory. This ansatz separates the usual Christoffel coefficients and the Yang-Mills gauge fields and leads to a standard Einstein general relativity coupled to the Yang-Mills theory with a particular gauge group. There are no new physical degrees of freedom in this formulation.

The formalism of matrix geometry is related to the algebra-valued tangent space formulation of Mann [7] and Wald [9]. In particular, Wald introduced the notion of an algebra-valued tangent space (more generally, algebra-valued tensor fields) and generalized the whole formalism of differential geometry (that is covariant derivatives, metrics, forms and curvature) to the algebra-valued case. This geometric interpretation was motivated by the preceding work of Cutler and Wald [8], where they found a *new type of gauge invariance* for a collection of *massless* spin-2 fields, more precisely, a consistent theory describing the interaction of a collection of massless spin-2 fields. The infinitesimal form of the new gauge transformations of the algebra-valued metric looks precisely like the infinitesimal diffeomorphisms but with *algebra-valued infinitesimal vector fields.* The consistency condition simply says that the commutator of two infinitesimal gauge transformations is an infinitesimal gauge transformation. The main conclusion of Cutler and Wald was that the consistency conditions can only be satisfied for *associative commutative algebras.* Therefore, all constructions "diagonalize" (everything commutes) and the interacting theory of a collection of massless spin-2 fields is simply a sum of the usual Einsten-Hilbert actions for each field (no cross-interactions). This restricts significantly the possible physical applications of this approach. Further, Wald noticed that the algebra-valued vector fields do not generate diffeomorphisms of a real manifold. Instead, they are diffeomorphisms on so-called 'algebra manifold', in which the *coordinates themselves are algebra-valued.* The supersymmetric extension of this approach was continued in [12].

Similar results concerning consistent cross-interactions of a collection of *massless* spin-2 fields were obtained in [15], where the multi-graviton theories were analyzed (from a different point of view). The authors of this work considered a collection of *massless* spin-2 tensor fields such that: i) the lagrangian contains no more than two derivatives of the fields, ii) the interactions can be continuously switched on, and iii) in the limit of no interaction, the lagrangian reduces to the sum of Pauli-Fierz lagrangians for each field. The authors of this paper consider the free limit condition described above to be the crucial one. They mention that Cutler and Wald did not analyze the extra conditions that must be imposed on the gauge symmetries coming from the free limit condition. They indicate that none of the examples of Cutler and Wald models have the correct free limit, since some of the gravitons come with the wrong sign and, therefore, the energy of the theory is unbounded from below. The free limit requirement gives an extra constraint, which implies that the algebra is not only commutative and associate but also *symmetric*. Therefore, the algebra has a trivial structure: it is simply the direct sum of one-dimensional ideals. This eliminates all cross-interactions. The only consistent deformation of the sum of Pauli-Fierz lagrangians is the sum of Einstein-Hilbert lagrangians (with a cosmological term) for each field.

In our previous papers [10, 11] we proposed to describe gravity by a multiplet of tensor fields with the corresponding gauge symmetry incorporated in the model. We would like to stress from the very beginning that our approach is different from the schemes studied by previous authors. Our main ingredient is the *matrix-valued two-tensor field*, so that the components of these tensor fields *do not commute* with each other, in general. Our algebra is associative but *noncommutative*. The other difference (related to the first one) is the form of the gauge transformations. We start from the very beginning with a real manifold with real-valued coordinates and the usual diffeomorphism group, so that there is no problem of defining the finite gauge transformations. Our gauge group is simply the product of diffeomorphisms and an internal group, say $\mathrm{U}(N)$. That is why we do not have the inconsistences studied by [8, 9] and [15] and we can allow our algebra to be non-commutative. In our approach there is a collection of tensor spin-2 fields, but *only one of them is massless*, all other fields are massive. How exactly this happens depends on the details of the symmetry breaking mechanism etc, but since we have only one (the usual real-valued) diffeomorphism group, only one field is massless. The parameter of our gauge transformations is a *real valued vector field*, not the algebra-valued vector fields of [8, 9, 15] needed to describe multiple *massless* spin-2 fields, which transform *independently* of each other. A similar approach to matrix gravity was proposed recently by Chamseddine [13], who considered an $\mathrm{SL}(2N,\mathbb{C}) \otimes \mathrm{SL}(2N,\mathbb{C})$ gauge theory with a spontaneous breakdown of symmetry giving one massless spin-2 field and $(2N^2-1)$ massive spin-2 fields.

Our idea was simply to repeat the Einstein's analysis of the causal structure of spacetime. We showed that the basic notions of general relativity are based on the geometrical interpretation of the wave equation that describes propagation of fields *without internal structure*, (in particular, light), that could transmit information in the spacetime. At the microscopic distances this role of the electromagnetic field could be played by other fields *with some internal structure*. That is why instead of a scalar wave equation we studied a *hyperbolic system* of linear second-order partial differential equations. This cardinally changes the standard geometric interpretation of General Relativity. Exactly in the same way as a scalar equation defines riemannian geometry, a system of wave equations generates a more general picture, that we call *Matrix Geometry*.

Morevover, we developed some kind of "matrix geometry". We defined the matrix-valued metric, affine connection, torsion and curvature. We have also found a particular form of a compatibility condition that enabled us to *explicitly* express the connection in terms of the derivatives of the metric. By using the matrix curvature we constructed an action functional that: i) contains no more than two derivatives of the fields in such a way that the equations of motion contain no more than two derivatives, ii) is invariant under both diffeomorphisms and the gauge transformations, iii) reduces to the sum of the Einstein-Hilbert action and the Yang-Mills action in the weak deformation limit. Notice that we do not obtain multiple Einstein-Hilbert actions, but just one. As we mentioned above there are several choices to make when generalizing the real-valued geometry to the matrix-valued one. In particular, the definition of the "measure", raising and lowering indices, definition of higher-order covariant derivatives, torsion constraints, and some others. As a result the action constructed in these papers is *not unique*.

In the present paper we propose an additional *physical principle* (*Low Energy Limit*) to construct an action functional that has all of the above properties but is also *unique* (or natural). We simply require that *the classical action of gravity is the semiclassical limit of the effective action of matter fields interacting with the gravitational field*. This is nothing but the ideology of "induced gravity". It is well known that the usual Einstein-Hilbert action (with a cosmological term) is just the first two terms of the asymptotic expansion of the effective action, or, simply, the first two coefficients of the asymptotic expansion of the trace of the heat kernel for a certain invariant second-order self-adjoint elliptic (in Euclidean formulation) partial differential operator with a *scalar* leading symbol given by the riemannian metric. That is why, in the present paper we propose to define the action of matrix gravity in a similar way as a linear combination of the first two coefficients of the trace of the heat kernel for a more general differential operator, with a *matrix-valued leading symbol* given by the matrix-valued metric. This way we avoid the constructions of matrix geometry, such as curvatures etc, which are non-unique. Although the language of differential geometry is certainly helpful, we simply do not need it here. In this approach *we get the invariants not from the curvatures but from the spectral invariants of a differential operator with a matrix-valued leading symbol*. This is the main idea and the main difference of the present paper from our previous papers [10, 11].

The outline of the present paper is as follows. In section 2.1 we introduce the necessary differential-geometric framework. We define the vector space of complex N-vectors and $N \times N$ matrices. In section 2.2 we introduce the fields as *densities*. This is done to avoid the definition of the measure, which we do not have since we do not have the metric. In section 2.3 the matrix-valued metric and the matrix-valued hamiltonian are introduced and certain non-degeneracy conditions are imposed. We consider two cases: elliptic and hyperbolic metric, depending on the eigenvalues of the hamiltonian. In section 2.4 it is shown that the above construction is closely related to a collection of Finsler geometries and the Finsler metrics are defined. In section 2.5 we consider matrix-valued p-forms and define the matrix-valued star operator $*$ (similar to the Hodge star operator). In order to define this operator in an invariant way we also introduce an additional ingredient: a matrix-valued measure. However, this measure is not related to the matrix-valued metric in the usual way (since it is not unique in the matrix case). Rather it is considered as an additional degree of freedom.

In section 3 we define the invariant differential operators of the first and the second order acting on matrix-valued p-forms. We define the exterior derivative, the coderivative as well as their covariant versions in section 3.1 and the gauge curvature in section 3.2. By using the covariant derivative and the coderivative we define invariant second-order differential operators (laplacians) in section 3.3 and invariant first-order differential operators (Dirac operators) in section 3.4. In section 3.5 we present the main operator studied in this paper in local coordinates. This is used later in the paper.

In section 4 we list several particular examples of decomposition of the general matrix valued metric and introduce the deformation parameter. We consider both commutative and noncommutative cases and compare them with the approaches of previous authors.

In section 5 we briefly summarize the heat kernel approach to compute the spectral asymptotics, the zeta-function regularization, and the asymptotic expansion of the effective action. The heat kernel coefficients introduced here play the crucial role in future development. In section 5.1 we compute the first two coefficients of the asymptotic expansion of the trace of the heat kernel for the non-Laplace type differential operator of general type (that is, a second-order operator with a non-scalar leading symbol). The Theorem 1 is one of the main results of the present paper. In section 5.2 we use the results of the Theorem 1 to calculate the coefficients A_0 and A_1 in the commutative (scalar) case and confirm the well known results. This serves as a 'consistency check' of the calculations of section 5.1 as well as shows how to make use of the general formulas of Theorem 1.

In section 6 the results for the heat kernel asymptotics are used to construct the action of "matrix gravity" via the induced gravity approach. The form of the action in the commutative limit is provided. It contains a metric and a scalar field (coming from the additional degree of freedom associated with the measure).

In section 7 the results are briefly discussed and a list of open interesting problems (singularities, high-energy behavior, dark energy, confinement etc) is presented.

2. Vector bundles

2.1 Vector spaces

To be precise, let V be a N-dimensional complex vector space with a hermitean inner product $\langle\ ,\ \rangle$. As usual, the dual vector space V^* is naturally identified with V, $\varphi \mapsto \bar\varphi$, by using the inner product:

$$\langle\psi,\varphi\rangle = \bar\psi(\varphi)\,, \tag{2.1}$$

where $\varphi, \psi \in V$ and $\bar{\psi} \in V^*$. If the elements φ of the vector space V are represented by N-dimensional contravariant vectors $\varphi = (\varphi^A)$, $(A = 1, \dots, N)$, and the elements of the dual space V^* by covariant vectors $\bar{\psi} = (\bar{\psi}_B)$, then the inner product is defined via the hermitean metric, $E = (E_{\dot{A}B})$,

$$\langle \psi, \varphi \rangle = \psi^\dagger E \varphi = \psi^{A*} E_{\dot{A}B} \varphi^B = \bar{\psi}_B \varphi^B \,, \tag{2.2}$$

where

$$E^\dagger = E \,, \qquad \bar{\psi} = \psi^\dagger E \,, \tag{2.3}$$

or

$$\bar{\psi}_B = \psi^{A*} E_{\dot{A}B} = (E_{\dot{B}A} \psi^A)^* \,. \tag{2.4}$$

Here and everywhere below we use the Einstein summation convention and follow the standard notation to mark the indices of the complex conjugated fields by dots; the symbol $*$ denotes the complex conjugation and the symbol $\dagger$ denotes the hermitean conjugation.

The vector space $\mathrm{End}(V)$ of endomorphisms of the vector space V is isomorphic to the vector space $\mathrm{Mat}(N, \mathbb{C})$ of complex $N \times N$ matrices $X = (X^A{}_B)$. The group of automorphisms $\mathrm{Aut}(V)$ of the vector space V is isomorphic to the general linear group $\mathrm{GL}(N, \mathbb{C})$ of complex nondegenerate $N \times N$ matrices. The adjoint $\bar{X}$ of an endomorphism X is defined by

$$\bar{X} = E^{-1} X^\dagger E \,, \tag{2.5}$$

so that for any $\varphi, \psi \in V$

$$\langle \psi, X\varphi \rangle = \langle \bar{X}\psi, \varphi \rangle \,. \tag{2.6}$$

The metric determines the subgroup of unitary endomorphisms (or isometries) G of the group of the automorphisms $\mathrm{Aut}(V)$ of the vector space V preserving the inner product, that is

$$G = \left\{ U \in \mathrm{GL}(N, \mathbb{C}) \mid \bar{U} U = \mathbb{I} \right\} , \tag{2.7}$$

where $\mathbb{I}$ is the identity endomorphism. The dimension of the group G is $\dim G = N^2$. In the case $E = \mathbb{I}$ the group G is nothing but $\mathrm{U}(N)$.

2.2 Vector bundles of densities

Now, let M be a smooth compact orientable n-dimensional manifold without boundary and $\mathcal{V}[w]$ be a smooth vector bundle of *densities* of weight w over the manifold M with the fiber V. Here and everywhere below we indicate explicitly the weight in the notation of the vector bundles. That is, the sections φ of the vector bundle $\mathcal{V}[w]$ are vector-valued functions $\varphi(x) = (\varphi^A(x))$ that transform under the diffeomorphisms $x'^\mu = x'^\mu(x)$ according to

$$\varphi'(x') = J^{-w}(x) \varphi(x) \,, \tag{2.8}$$

where

$$J(x) = \det \left[\frac{\partial x'^\mu(x)}{\partial x^\alpha} \right] . \tag{2.9}$$

Hereafter we label the local coordinates on the manifold M by Greek indices which run over $0, 1, \ldots, n-1$. The small Latin indices will be used to label the space coordinates, i.e. the local coordinates on hypersurfaces transversal to the time coordinate; they will run over $1, 2, \ldots, n-1$.

Further, we denote by $\mathcal{V}^*[w]$ the dual bundle of densities of weight w with the fiber V^* and by $\mathrm{End}(\mathcal{V})[w]$ the bundle of endomorphisms-valued densities of weight w of the bundle $\mathcal{V}[w]$ with elements being matrix-valued functions $U = (U^A{}_B(x))$.

We will also consider the bundle of vector-valued antisymmetric densities of weight w of type $(0,p)$ (p-forms), $\Lambda_p[w] = ((\wedge^p T^*M) \otimes \mathcal{V})[w]$, and the bundle of anti-symmetric densities of weight w of type $(p,0)$, $\Lambda^p[w] = ((\wedge^p TM) \otimes \mathcal{V})[w]$. Finally, we also introduce the following notation $E_p[w] = ((\wedge^p T^*M) \otimes \mathrm{End}(\mathcal{V}))[w]$ for the bundle of endomorphism-valued densities of weight w of type $(0,p)$ and $E^p[w] = ((\wedge^p TM) \otimes \mathrm{End}(\mathcal{V}))[w]$ for the bundle of endomorphism-valued densities of weight w of type $(p,0)$.

The group G is also promoted to a smooth vector bundle $\mathcal{G}[0]$ (gauge group), a sub-bundle of the bundle $\mathrm{End}(\mathcal{V})[0]$. Under the action of the gauge group $\mathcal{G}[0]$ the sections φ of the bundles $\mathcal{V}[w]$ and $\mathcal{V}^*[w]$ transform as

$$\varphi'(x) = U(x)\varphi(x)\,, \qquad \bar{\varphi}'(x) = \bar{\varphi}(x)U^{-1}(x)\,, \tag{2.10}$$

The metric E is assumed to be a section of the bundle $(\mathcal{V}^* \otimes \mathcal{V}^*)[\alpha]$ with some weight α. It is invariant under the gauge transformations

$$E'(x) = (U^\dagger(x))^{-1} E(x) U^{-1}(x) = E(x)\,, \tag{2.11}$$

which guarantees the invariance of the fiber inner product.

However, to get a diffeomorphism-invariant L^2 inner product

$$(\psi, \varphi) = \int\limits_M dx\ \langle \psi(x), \varphi(x)\rangle\,, \tag{2.12}$$

and the L^2 norm

$$||\varphi||^2 = (\varphi, \varphi) = \int\limits_M dx\ \langle \varphi(x), \varphi(x)\rangle\,, \tag{2.13}$$

on the vector bundle $\mathcal{V}[w]$ with the standard Lebesgue measure $dx = dx^1 \wedge \cdots \wedge dx^n$, the metric must be a *density* of weight $(1-2w)$, more precisely, a section of the bundle $(\mathcal{V}^* \otimes \mathcal{V}^*)[1-2w]$. Then the completion of $C^\infty(\mathcal{V}[w])$ in this norm defines the Hilbert space $L^2(\mathcal{V}[w])$. Notice that this means, that on the bundle $\mathcal{V}[\frac{1}{2}]$ of densities of weight $w = \frac{1}{2}$ the metric $E(x)$ is invariant under the diffeomorphisms as well, and, therefore, without loss of generality can be assumed to be *constant*.

2.3 Noncommutative metric

Our goal is to construct covariant self-adjoint second order differential operators acting on a smooth sections of the bundles $\Lambda_p[\frac{1}{2}]$ and $\Lambda^p[\frac{1}{2}]$, that are covariant under both diffeomorphisms,

$$L'\varphi'(x') = J^{-1/2}(x) L\varphi(x)\,, \tag{2.14}$$

and the gauge transformations

$$L'\varphi'(x) = U(x)L\varphi(x)\,. \tag{2.15}$$

To do this we need to define the following objects. First of all, we need an isomorphism between the bundles $\Lambda_p[\frac{1}{2}]$ and $\Lambda^p[\frac{1}{2}]$. This is usually achieved by the Hodge star operator, which is defined with the help of a riemannian metric. This is exactly the place where we want to *generalize the standard theory* since we do not want to introduce a riemannian metric; instead of a riemannian metric (which is an isomorphism between the tangent TM and the cotangent T^*M bundles) we introduce an *isomorphism* between the bundles $\Lambda_1[w] = (T^*M \otimes \mathcal{V})[w]$ and $\Lambda^1[w] = (TM \otimes \mathcal{V})[w]$, i.e.

$$a : \Lambda_1[w] \to \Lambda^1[w]\,. \tag{2.16}$$

Such an isomorphism is determined by a section of the vector bundle $(TM \otimes TM \otimes \mathrm{End}(\mathcal{V}))[0]$ defined by the matrix-valued symmetric tensor $a^{\mu\nu} = \left(a^{\mu\nu A}{}_B(x)\right)$ that satisfies the following conditions:

1. The matrix $a^{\mu\nu}$ is self-adjoint (recall that $\overline{a^{\mu\nu}} = E^{-1}(a^{\mu\nu})^\dagger E$),

$$\overline{a^{\mu\nu}} = a^{\mu\nu} \tag{2.17}$$

   and symmetric,

$$a^{\nu\mu} = a^{\nu\mu}\,. \tag{2.18}$$

2. It transforms under the diffeomorphisms as

$$a'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}a^{\alpha\beta}(x)\,, \tag{2.19}$$

   and under the gauge transformations as

$$a'^{\mu\nu}(x) = U(x)a^{\mu\nu}(x)U^{-1}(x)\,. \tag{2.20}$$

3. Consider the matrix

$$H(x,\xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu\,, \tag{2.21}$$

   with ξ being a cotangent vector ξ at a spacetime point x. Since this matrix is self-adjoint, all its eigenvalues $h_{(i)}(x,\xi)$, $(i = 1,\dots,s)$, must be real. We will assume that they have constant multiplicities d_i. The eigenvalues are invariant under the gauge transformations (2.10) and transform under the diffeomorphisms as

$$h'_i(x',\xi) = h_i(x,\xi')\,, \tag{2.22}$$

   where

$$\xi'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\xi_\alpha\,. \tag{2.23}$$

   We will also require that this matrix satisfies one of the following conditions. *Elliptic case.* The matrix $H(x,\xi)$ is positive definite, i.e. all its eigenvalues are positive, $h_{(i)}(x,\xi) > 0$, for any x and any $\xi \neq 0$. *Hyperbolic case.* There is a one-form

$\tau = \tau(x)$ (specifying the time direction) such that at each point x for each eigenvalue $h_{(i)}(x, \tau(x)) < 0$, and for any cotangent vector $\zeta \neq 0$ not parallel to τ each equation $h_{(i)}(x, \zeta + \lambda\tau(x)) = 0$, has exactly two real distinct roots, $\lambda = \lambda_\pm(x, \zeta)$. More precisely, a second-order partial differential operator L with the principal symbol $H(x, \xi)$ is *hyperbolic* if the roots $\lambda_j(x, \zeta)$ of the characteristic equation

$$\det H(x, \zeta + \lambda\tau(x)) = 0 \tag{2.24}$$

are real and *strictly hyperbolic* if they are real and distinct [25, 29]).

The infinitesimal forms of the diffeomorphisms and the gauge transformations are obtained as follows. Let $x' = f_t(x)$ be a one-parameter subgroup of diffeomorphisms such that

$$f_t(x)|_{t=0} = x, \qquad \text{and} \qquad \xi(x) = \frac{d}{dt} f_t(x)\Big|_{t=0} . \tag{2.25}$$

Then

$$\delta_\xi a^{\mu\nu} \equiv \frac{d}{dt} a'^{\mu\nu}(x')\Big|_{t=0} = \xi^\alpha \partial_\alpha a^{\mu\nu} - (\partial_\alpha \xi^\mu) a^{\alpha\nu} - (\partial_\alpha \xi^\nu) a^{\mu\alpha} . \tag{2.26}$$

Similarly, let U_t be a one-parameter subgroup of the gauge group such that

$$U_t|_{t=0} = \mathbb{I} , \qquad \text{and} \qquad \omega = \frac{d}{dt} U_t\Big|_{t=0} . \tag{2.27}$$

Then

$$\delta_\omega a^{\mu\nu} \equiv \frac{d}{dt} a'^{\mu\nu}\Big|_{t=0} = [\omega, a^{\mu\nu}] . \tag{2.28}$$

Notice that these infinitesimal transformations are different from the gauge transformations studied in [8, 9, 15] since there is only one vector field ξ.

2.4 Finsler geometry

The above construction is closely related to Finsler geometry [28]. First of all, we note that the eigenvalues $h_{(i)}(x, \xi)$ are homogeneous functions of ξ of degree 2, i.e.

$$h_{(i)}(x, \lambda\xi) = \lambda^2 h_{(i)}(x, \xi) . \tag{2.29}$$

Next, we define the *Finsler metrics*

$$g^{\mu\nu}_{(i)}(x, \xi) = \frac{1}{2} \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} h_{(i)}(x, \xi) . \tag{2.30}$$

All these metrics are non-degenerate. In the elliptic case all metrics $g^{\mu\nu}_{(i)}$ are positive definite; in the hyperbolic case they have the signature $(- + \cdots +)$. In the case when a Finsler metric does not depend on ξ it is simply a riemannian metric.

The Finsler metrics are homogeneous functions of ξ of degree 0

$$g^{\mu\nu}_{(i)}(x, \lambda\xi) = g^{\mu\nu}_{(i)}(x, \xi) , \tag{2.31}$$

so that they depend only on the direction of the covector ξ but not on its magnitude. This leads to a number of identities, in particular,

$$h_{(i)}(x,\xi) = g^{\mu\nu}_{(i)}(x,\xi)\xi_\mu\xi_\nu \tag{2.32}$$

and

$$\frac{1}{2}\frac{\partial}{\partial\xi_\mu}h_{(i)}(x,\xi) = g^{\mu\nu}_{(i)}(x,\xi)\xi_\nu\,. \tag{2.33}$$

Next we define the inverse (covariant) Finsler metrics by

$$g_{(i)\mu\nu}(x,\dot{x})g^{\nu\alpha}_{(i)}(x,\xi) = \delta^\alpha_\mu\,, \tag{2.34}$$

where $\dot{x}^\mu$ is the tangent vector defined by

$$\dot{x}^\mu = g^{\mu\nu}_{(i)}(x,\xi)\xi_\nu\,, \tag{2.35}$$

so that

$$\xi_\mu = g_{(i)\mu\nu}(x,\dot{x})\dot{x}^\nu\,. \tag{2.36}$$

Finally, this enables one to define the Finsler intervals

$$ds^2_{(i)} = g_{(i)\mu\nu}(x,\dot{x})dx^\mu dx^\nu\,. \tag{2.37}$$

The existense of the Finsler metrics allows one to define various connections, curvatures etc (for details see [28]).

As we see, the propagation of gauge fields induces Finsler geometry.

2.5 Star operators

Since the map a (2.16) is an isomorphism, the inverse map

$$b = a^{-1} : \Lambda^1[w] \to \Lambda_1[w]\,, \tag{2.38}$$

is well defined. In other words, for any $\psi^\mu = (\psi^{\mu A})$ there is a unique $\varphi_\nu = (\varphi^A_\nu)$ satisfying the equation $a^{\mu\nu}\varphi_\nu = \psi^\mu$, and, therefore, there is a unique solution of the equations

$$a^{\mu\nu}b_{\nu\alpha} = \delta^\mu_\alpha\,, \qquad b_{\alpha\nu}a^{\nu\mu} = \delta^\mu_\alpha\,. \tag{2.39}$$

Notice that the matrix $b_{\mu\nu}$ has the property

$$\bar{b}_{\mu\nu} = b_{\nu\mu}\,, \tag{2.40}$$

but is neither symmetric $b_{\mu\nu} \neq b_{\nu\mu}$ nor self-adjoint $\bar{b}_{\mu\nu} \neq b_{\mu\nu}$.

The isomorphism a naturally defines the maps between the bundles $\Lambda_p[w]$ and $\Lambda^p[w]$

$$A : \Lambda_p[w] \to \Lambda^p[w]\,, \qquad B : \Lambda^p[w] \to \Lambda_p[w]\,, \tag{2.41}$$

as follows

$$(A\varphi)^{\mu_1\cdots\mu_p} = A^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}\varphi_{\nu_1\cdots\nu_p}\,, \tag{2.42}$$

where

$$A^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p} = \mathrm{Alt}_{\mu_1\cdots\mu_p}\,\mathrm{Alt}_{\nu_1\cdots\nu_p}\,a^{\mu_1\nu_1}\cdots a^{\mu_p\nu_p} \tag{2.43}$$

and

$$(B\varphi)_{\mu_1\cdots\mu_p} = B_{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}\varphi^{\nu_1\cdots\nu_p}\,, \tag{2.44}$$

where

$$B_{\mu_1\cdots\mu_p\nu_1\cdots\nu_p} = \mathrm{Alt}_{\mu_1\cdots\mu_p}\,\mathrm{Alt}_{\nu_1\cdots\nu_p}\,b_{\mu_1\nu_1}\cdots b_{\mu_p\nu_p} \tag{2.45}$$

Here $\mathrm{Alt}_{\mu_1\cdots\mu_p}$ denotes the complete antisymmetrization over the indices $\mu_1,\ldots,\mu_p$.

We will assume that these maps are isomorphisms as well. Strictly speaking, one has to prove this. This is certainly true for the weakly deformed maps (maps close to the identity; more on this later). Then the inverse operator

$$A^{-1} : \Lambda^p[w] \to \Lambda_p[w]\,, \tag{2.46}$$

is defined by

$$(A^{-1}\varphi)_{\mu_1\cdots\mu_p} = A^{-1}_{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}\varphi^{\nu_1\cdots\nu_p}\,, \tag{2.47}$$

where A^{-1} is determined by the equation

$$A^{-1}_{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}A^{\nu_1\cdots\nu_p\alpha_1\cdots\alpha_p} = \delta^{\alpha_1}_{[\mu_1}\cdots\delta^{\alpha_p}_{\mu_p]}\,. \tag{2.48}$$

Notice that because of the noncommutativity, the inverse operator A^{-1} is not equal to the operator B, so that $A^{-1}B \neq \mathrm{Id}$.

This is used further to define the natural fiber inner product on the space of p-forms Λ_p via

$$\langle\psi,\varphi\rangle = \frac{1}{p!}\bar\psi_{\mu_1\cdots\mu_p}A^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}\varphi_{\nu_1\cdots\nu_p}\,. \tag{2.49}$$

We will also need a smooth self-adjoint section ρ (given locally by the matrix-valued function $\rho = (\rho^A{}_B(x))$) of the bundle $\mathrm{End}(\mathcal{V})[\frac{1}{2}]$ of endomorphism-valued densities of weight $\frac{1}{2}$. That is ρ satisfies the equation

$$\bar\rho = \rho\,, \tag{2.50}$$

(recall that $\bar\rho = E^{-1}\rho^\dagger E$), and transforms under diffeomorphisms as

$$\rho'(x') = J^{-1/2}(x)\rho(x)\,, \tag{2.51}$$

and under the action of the gauge group $\mathcal{G}[0]$ as

$$\rho'(x) = U(x)\rho(x)U^{-1}(x)\,. \tag{2.52}$$

Clearly, the matrices

$$a^{\mu\nu} = g^{\mu\nu}\mathbb{I}\,, \qquad \rho = g^{1/4}\mathbb{I}\,, \tag{2.53}$$

where $g^{\mu\nu}$ is a (pseudo)-riemannian metric and

$$g = |\det g^{\mu\nu}|^{-1}\,, \tag{2.54}$$

satisfy all the above conditions. We will refer to this particular case as the *commutative limit.*

Of course, (on orientable manifolds) we always have the standard volume form ε, which is a section of the bundle $E_n[-1]$ given by the completely antisymmetric Levi-Civita symbol $\varepsilon_{\mu_1\cdots\mu_n}$. The contravariant Levi-Civita symbol $\tilde{\varepsilon}$ with components

$$\varepsilon^{\mu_1\cdots\mu_n} = \sigma\varepsilon_{\mu_1\cdots\mu_n}\,, \tag{2.55}$$

with $\sigma = +1$ in the elliptic case and $\sigma = -1$ in the hyperbolic case, is a section of the bundle $E^n[1]$.

These densities are used to define the standard isomorphisms between the densities bundles

$$\varepsilon : \Lambda^p[w] \to \Lambda_{n-p}[w-1]\,, \qquad \tilde{\varepsilon} : \Lambda_p[w] \to \Lambda^{n-p}[w+1] \tag{2.56}$$

by

$$(\varepsilon\varphi)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\varphi^{\nu_1\cdots\nu_p}\,, \quad (\tilde{\varepsilon}\varphi)^{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\varepsilon^{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\varphi_{\nu_1\cdots\nu_p}\,. \tag{2.57}$$

By using the well known identity

$$\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\varepsilon^{\mu_1\cdots\mu_{n-p}\lambda_1\cdots\lambda_p} = \sigma(n-p)!p!\delta^{\lambda_1}_{[\nu_1}\cdots\delta^{\lambda_p}_{\nu_p]} \tag{2.58}$$

we get

$$\tilde{\varepsilon}\varepsilon = \varepsilon\tilde{\varepsilon} = \sigma(-1)^{p(n-p)}\,\mathrm{Id}\,. \tag{2.59}$$

By combining ε and $\tilde{\varepsilon}$ with the endomorphism ρ we can get the invariant form $\varepsilon\rho^2$, which is a section of the bundle $E_n[0]$, and the form $\tilde{\varepsilon}\rho^{-2}$, a section of the bundle $E^n[0]$. Notice, however, that, in general, the contravariant form $\tilde{\varepsilon}\rho^{-2}$ is not equal to that obtained by raising indices of the covariant form $\varepsilon\rho^2$, i.e. $\tilde{\varepsilon}\rho^{-2} \neq A\varepsilon\rho^2$ or

$$\varepsilon^{\mu_1\cdots\mu_n}\rho^{-2} \neq A^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\varepsilon_{\nu_1\cdots\nu_n}\rho^2\,. \tag{2.60}$$

If we require this to be the case then the matrix ρ should be defined by

$$\rho = \eta^{-1/4}\,, \tag{2.61}$$

where

$$\eta = \sigma\frac{1}{n!}\varepsilon_{\mu_1\cdots\mu_n}\varepsilon_{\nu_1\cdots\nu_n}a^{\mu_1\nu_1}\cdots a^{\mu_n\nu_n}\,. \tag{2.62}$$

Clearly, η is a density of weight (-2). Since $a^{\mu\nu}$ is self-adjoint, we also find that η and, hence, ρ is self-adjoint. The problem is that in general η is not positive definite. Notice that in the commutative limit

$$\eta = \sigma\,\det g^{\mu\nu} = |\det g_{\mu\nu}|^{-1}\,, \tag{2.63}$$

which is strictly positive.

Therefore, we can finally define two different star operators

$$*, \tilde{*} : \Lambda_p[w] \to \Lambda_{n-p}[w] \tag{2.64}$$

by

$$* = \varepsilon\rho A\rho\,, \qquad \tilde{*} = \rho^{-1}A^{-1}\rho^{-1}\tilde{\varepsilon} \tag{2.65}$$

that is

$$(*\varphi)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\rho A^{\nu_1\cdots\nu_p\alpha_1\cdots\alpha_p}\rho\varphi_{\alpha_1\cdots\alpha_p}\,, \tag{2.66}$$

$$(\tilde{*}\varphi)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\rho^{-1}(A^{-1})_{\mu_1\cdots\mu_{n-p}\beta_1\cdots\beta_{n-p}}\rho^{-1}\varepsilon^{\beta_1\cdots\beta_{n-p}\alpha_1\cdots\alpha_p}\varphi_{\alpha_1\cdots\alpha_p}\,. \tag{2.67}$$

The star operators are self-adjoint in the sense

$$\langle\varphi, *\psi\rangle = \langle *\varphi, \psi\rangle\ , \qquad \langle\varphi, \tilde{*}\psi\rangle = \langle\tilde{*}\varphi, \psi\rangle\ , \tag{2.68}$$

and satisfy the relation: for any p form

$$*\tilde{*} = \tilde{*}* = \sigma(-1)^{p(n-p)}\,\mathrm{Id}\ . \tag{2.69}$$

2.6 Hilbert spaces

Now, let us consider the bundles of densities of weight $\frac{1}{2}$, $\mathcal{V}[\frac{1}{2}]$, and, more generally, $\Lambda_p[\frac{1}{2}]$. If $dx = dx^1 \wedge \cdots \wedge dx^n$ is the standard Lebesgue measure in a local chart on M, then we define the diffeomorphism-invariant L^2-inner product

$$(\psi, \varphi) = \int\limits_M dx\ \langle\psi(x), \varphi(x)\rangle\ , \tag{2.70}$$

and the L^2 norm

$$||\varphi||^2 = (\varphi, \varphi) = \int\limits_M dx\ \langle\varphi(x), \varphi(x)\rangle\,. \tag{2.71}$$

The completion of $C^\infty(\Lambda_p[\frac{1}{2}])$ in this norm defines the Hilbert space $L^2(\Lambda_p[\frac{1}{2}])$.

3. Differential operators

3.1 Noncommutative exterior calculus

Next, we define the invariant differential operators on smooth sections of the bundles $\Lambda_p[0]$ and $\Lambda^p[1]$. The exterior derivative (the *gradient*)

$$d : C^\infty(\Lambda_p[0]) \to C^\infty(\Lambda_{p+1}[0]) \tag{3.1}$$

is defined by

$$(d\varphi)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\varphi_{\mu_2\cdots\mu_p]}\,, \qquad \text{if } p = 0, \ldots, n-1\,, \tag{3.2}$$

$$d\varphi = 0 \qquad \text{if } p = n\,, \tag{3.3}$$

where the square brackets denote the complete antisymmetrization. The coderivative (the *divergence*)

$$\tilde{d} : C^\infty(\Lambda^p[1]) \to C^\infty(\Lambda^{p-1}[1]) \tag{3.4}$$

is defined by

$$\tilde{d} = \sigma(-1)^{np+1}\tilde{\varepsilon} d\varepsilon \,. \tag{3.5}$$

By using (2.58) one can easily find

$$(\tilde{d}\varphi)^{\mu_1\cdots\mu_{p-1}} = \partial_\mu \varphi^{\mu\mu_1\cdots\mu_{p-1}} \qquad \text{if } p = 1, \ldots, n \,, \tag{3.6}$$

$$\tilde{d}\varphi = 0 \qquad \text{if } p = 0 \,. \tag{3.7}$$

One can also show that these definitions are covariant and satisfy the standard relations

$$d^2 = \tilde{d}^2 = 0 \,. \tag{3.8}$$

The endomorphism ρ is a section of the bundle $\mathrm{End}(\mathcal{V})[\frac{1}{2}]$. Therefore, if φ is a section of the bundle $\Lambda_p[\frac{1}{2}]$, the quantity $\rho^{-1}\varphi$ is a section of the bundle $\Lambda_p[0]$. Hence, the derivative $d(\rho^{-1}\varphi)$ is well defined as a smooth section of the vector bundle $\Lambda_{p+1}[0]$. By scaling back with the factor ρ we get an invariant differential operator

$$\rho d\rho^{-1} : \; C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda_{p+1}\left[\frac{1}{2}\right]\right) . \tag{3.9}$$

Similarly, we can define the invariant operator of codifferentiation on densities of weight $\frac{1}{2}$

$$\rho^{-1}\tilde{d}\rho : \; C^\infty\left(\Lambda^p\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda^{p-1}\left[\frac{1}{2}\right]\right) . \tag{3.10}$$

Now, let $\mathcal{B}$ be a smooth anti-self-adjoint section of the vector bundle $E_1[0]$, defined by the matrix-valued covector $\mathcal{B}_\mu = (\mathcal{B}_\mu{}^A{}_B(x))$ i.e.

$$\bar{\mathcal{B}}_\mu = -\mathcal{B}_\mu \,, \tag{3.11}$$

that transforms under diffeomorphisms as

$$\mathcal{B}'_\mu(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\mathcal{B}_\alpha(x) \,, \tag{3.12}$$

and under the gauge transformations as

$$\mathcal{B}'_\mu(x) = U(x)\mathcal{B}_\mu(x)U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x) \,. \tag{3.13}$$

Such a section naturally defines the maps:

$$\mathcal{B} : \Lambda_p[w] \to \Lambda_{p+1}[w] \tag{3.14}$$

by

$$(\mathcal{B}\varphi)_{\mu_1\cdots\mu_{p+1}} = (p+1)\mathcal{B}_{[\mu_1}\varphi_{\mu_2\cdots\mu_{p+1}]} \tag{3.15}$$

and

$$\tilde{\mathcal{B}} : \Lambda^p[w] \to \Lambda^{p-1}[w] \tag{3.16}$$

by

$$(\tilde{\mathcal{B}}\varphi)^{\mu_1\cdots\mu_{p-1}} = \mathcal{B}_\mu \varphi^{\mu\mu_1\cdots\mu_{p-1}} \,. \tag{3.17}$$

Notice that

$$\tilde{\mathcal{B}} = \sigma(-1)^{np+1}\tilde{\varepsilon}\mathcal{B}\varepsilon \tag{3.18}$$

similar to (3.5).

This enables us to define the covariant exterior derivative

$$\mathcal{D} : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda_{p+1}\left[\frac{1}{2}\right]\right). \tag{3.19}$$

by

$$\mathcal{D} = \rho(d+\mathcal{B})\rho^{-1} \tag{3.20}$$

and the covariant coderivative

$$\tilde{\mathcal{D}} : C^\infty\left(\Lambda^p\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda^{p-1}\left[\frac{1}{2}\right]\right), \tag{3.21}$$

by

$$\tilde{\mathcal{D}} = \sigma(-1)^{np+1}\tilde{\varepsilon}\mathcal{D}\varepsilon = \rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho\,. \tag{3.22}$$

These operators transform covariantly under both the diffeomorphisms and the gauge transformations.

3.2 Noncommutative gauge curvature

One can easily show that the square of the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$

$$\mathcal{D}^2 : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda_{p+2}\left[\frac{1}{2}\right]\right) \tag{3.23}$$

$$\tilde{\mathcal{D}}^2 : C^\infty\left(\Lambda^{p+2}\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda^{p}\left[\frac{1}{2}\right]\right) \tag{3.24}$$

are zero-order differential operators. In particular, in the case $p = 0$ they define the gauge curvature $\mathcal{F}$, which is a section of the bundle $E_2[0]$, by

$$(\mathcal{D}^2\varphi)_{\mu\nu} = \rho\mathcal{F}_{\mu\nu}\rho^{-1}\varphi\,, \qquad \tilde{\mathcal{D}}^2\varphi = \rho^{-1}\mathcal{F}_{\mu\nu}\rho\varphi^{\nu\mu}\,, \tag{3.25}$$

where

$$\mathcal{F} = d\mathcal{B} + [\mathcal{B},\mathcal{B}]\,, \tag{3.26}$$

and the brackets $[\,,\,]$ denote the Lie bracket of two matrix-valued 1-forms, i.e.

$$[A,B]_{\mu\nu} = A_\mu B_\nu - B_\nu A_\mu\,. \tag{3.27}$$

The gauge curvature is anti-self-adjoint

$$\bar{\mathcal{F}}_{\mu\nu} = -\mathcal{F}_{\mu\nu} \tag{3.28}$$

and transforms covariantly under diffeomorphisms

$$\mathcal{F}'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\mathcal{F}_{\alpha\beta}(x)\,, \tag{3.29}$$

and under the gauge transformations

$$\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^{-1}(x)\,. \tag{3.30}$$

3.3 Noncommutative laplacians

Finally, by using the objects introduced above we can define second-order differential operators that are covariant under both diffeomorphisms, and the gauge transformations. In order to do that we need first-order differential operators (divergences)

$$\bar{\mathcal{D}}:\ C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right)\to C^\infty\left(\Lambda_{p-1}\left[\frac{1}{2}\right]\right), \tag{3.31}$$

First of all, by using the L^2 inner product on the bundle $\Lambda_p[\frac{1}{2}]$ (recall that the metric E is constant in this case) we define the adjoint operator $\bar{\mathcal{D}}$ by

$$\langle\varphi,\mathcal{D}\psi\rangle=\langle\bar{\mathcal{D}}\varphi,\psi\rangle\,. \tag{3.32}$$

This gives

$$\bar{\mathcal{D}}=-A^{-1}\tilde{\mathcal{D}}A=-\sigma(-1)^{np+1}A^{-1}\tilde{\varepsilon}\mathcal{D}\varepsilon A=-A^{-1}\rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho A\,, \tag{3.33}$$

which in local coordinates reads

$$(\bar{\mathcal{D}}\varphi)_{\mu_1\cdots\mu_p}=-(A^{-1})_{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}\rho^{-1}(\partial_\nu+\mathcal{B}_\nu)\rho A^{\nu\nu_1\cdots\nu_p\alpha_1\cdots\alpha_{p+1}}\varphi_{\alpha_1\cdots\alpha_{p+1}}\,. \tag{3.34}$$

The problem with this definition is that usually it is difficult to find the matrix $(A^{-1})_{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}$.

Then we define the second order operators

$$\bar{\mathcal{D}}\mathcal{D}\,,\mathcal{D}\bar{\mathcal{D}},\Delta:\ C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right)\to C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right), \tag{3.35}$$

where the "noncommutative laplacian" is

$$\begin{aligned}\Delta &= -\bar{\mathcal{D}}\mathcal{D}-\mathcal{D}\bar{\mathcal{D}}\\ &= A^{-1}\tilde{\mathcal{D}}A\mathcal{D}+\mathcal{D}A^{-1}\tilde{\mathcal{D}}A\\ &= A^{-1}\rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho A\rho(d+\mathcal{B})\rho^{-1}+\rho(d+\mathcal{B})\rho^{-1}A^{-1}\rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho A\,.\end{aligned} \tag{3.36}$$

In local coordinates this reads

$$\begin{aligned}(\Delta\varphi)_{\mu_1\cdots\mu_p} = \Big\{&(p+1)A^{-1}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_p}\rho^{-1}(\partial_\nu+\mathcal{B}_\nu)\rho\times\\ &\times A^{\nu\nu_1\ldots\nu_p\alpha\alpha_1\ldots\alpha_p}\rho(\partial_\alpha+\mathcal{B}_\alpha)\rho^{-1}+\\ &+\rho(\partial_{[\mu_1}+\mathcal{B}_{[\mu_1})\rho^{-1}A^{-1}_{\mu_2\ldots\mu_{p-1}]\nu_1\ldots\nu_{p-1}}\times\\ &\times\rho^{-1}(\partial_\nu+\mathcal{B}_\nu)\rho A^{\nu\nu_1\ldots\nu_{p-1}\alpha_1\ldots\alpha_p}\Big\}\varphi_{\alpha_1\ldots\alpha_p}\,.\end{aligned} \tag{3.37}$$

We could have also defined the coderivatives by

$$\bar{\mathcal{D}}_1=-*\mathcal{D}*\,,\qquad\bar{\mathcal{D}}_2=-B\tilde{\mathcal{D}}A\,,\qquad\bar{\mathcal{D}}_3=-\tilde{*}\mathcal{D}*\,,\qquad\bar{\mathcal{D}}_4=-*\mathcal{D}\tilde{*}\,. \tag{3.38}$$

These operators have the advantage that $\bar{\mathcal{D}}_1$ is polynomial in the matrix $a^{\mu\nu}$ and $\bar{\mathcal{D}}_2$ is polynomial in the matrices $a^{\mu\nu}$ and $b_{\mu\nu}$. However, the second order operators $\bar{\mathcal{D}}_j\mathcal{D}$, $\mathcal{D}\bar{\mathcal{D}}_j$ and $\Delta_j = -\bar{\mathcal{D}}_j\mathcal{D} - \mathcal{D}\bar{\mathcal{D}}_j$, $(j = 1, 2, 3, 4)$, are not self-adjoint, in general. In the commutative limit all these definitions coincide with the standard de Rham laplacian.

In the special case $p = 0$ the "noncommutative laplacian" Δ reads

$$\Delta = \rho^{-1}(\tilde{d} + \tilde{\mathcal{B}})\rho A \rho (d + \mathcal{B})\rho^{-1} , \tag{3.39}$$

which in local coordinates has the form

$$\Delta = \rho^{-1}(\partial_\mu + \mathcal{B}_\mu)\rho a^{\mu\nu}\rho(\partial_\nu + \mathcal{B}_\nu)\rho^{-1} . \tag{3.40}$$

3.4 Noncommutative Dirac operator

It is worth mentioning another approach. Suppose we are given a self-adjoint section Γ of the bundle $E_1[0]$, given locally by matrix valued vector $\Gamma^\mu = (\Gamma^{\mu A}{}_B(x))$ satisfying

$$\bar{\Gamma}^\mu = \Gamma^\mu , \tag{3.41}$$

where as usual $\bar{\Gamma}^\mu = E^{-1}(\Gamma^\mu)^\dagger E$. It transforms under the diffeomorphisms as

$$\Gamma'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\Gamma^\alpha(x) \tag{3.42}$$

and under the gauge transformation as

$$\Gamma'^\mu(x) = U(x)\Gamma^\mu(x)U^{-1}(x) . \tag{3.43}$$

Then the matrix

$$a^{\mu\nu} = \frac{1}{2}\left(\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu\right) \tag{3.44}$$

is self-adjoint $\overline{a^{\mu\nu}} = a^{\mu\nu}$ and symmetric $a^{\mu\nu} = a^{\nu\mu}$. Moreover, suppose this matrix satisfies all the conditions for the matrix a listed above in section 2.3. Most importantly, the matrix

$$H(x, \xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu = [\Gamma(x, \xi)]^2 , \tag{3.45}$$

where

$$\Gamma(x, \xi) = \Gamma^\mu(x)\xi_\mu , \tag{3.46}$$

is nondegenerate and satisfies the ellipticity or hyperbolicity conditions formulated in section 2.3. (The advantage of this approach is that in the elliptic case the matrix $H(x, \xi)$ is automatically positive definite.)

Such vector naturally defines the map

$$\Gamma : \; C^\infty(\Lambda^p[w]) \to C^\infty(\Lambda^{p+1}[w]) \tag{3.47}$$

by

$$(\Gamma\varphi)^{\mu_1\cdots\mu_{p+1}} = (p + 1)\Gamma^{[\mu_1}\varphi^{\mu_2\cdots\mu_{p+1}]} \tag{3.48}$$

and the map

$$\tilde{\Gamma} : \; C^\infty(\Lambda_p[w]) \to C^\infty(\Lambda_{p-1}[w]) \tag{3.49}$$

by

$$(\tilde{\Gamma}\varphi)_{\mu_1\dots\mu_{p-1}} = \Gamma^\mu \varphi_{\mu\mu_1\dots\mu_p}\,. \tag{3.50}$$

Therefore, we can define first-order invariant differential operator ("noncommutative Dirac operator")

$$D : C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \to C^\infty\left(\Lambda_p\left[\frac{1}{2}\right]\right) \tag{3.51}$$

by

$$D = \tilde{\Gamma}\mathcal{D} = \tilde{\Gamma}\rho(d+\mathcal{B})\rho^{-1}\,, \tag{3.52}$$

which in local coordinates reads

$$(D\varphi)_{\mu_1\dots\mu_p} = (p+1)\Gamma^\mu\rho(\partial_{[\mu}+\mathcal{B}_{[\mu})\rho^{-1}\varphi_{\mu_1\dots\mu_p]}\,. \tag{3.53}$$

The adjoint of this operator with respect to the L^2 inner product is

$$\bar{D} = -A^{-1}\tilde{\mathcal{D}}\Gamma A = -A^{-1}\rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho\Gamma A\,, \tag{3.54}$$

which in local coordinates becomes

$$(\bar{D}\varphi)_{\mu_1\dots\mu_p} = -(p+1)A^{-1}_{\mu_1\dots\mu_p\nu_1\dots\nu_p}\rho^{-1}(\partial_\nu+\mathcal{B}_\nu)\rho\Gamma^{[\nu}A^{\nu_1\dots\nu_p]\alpha_1\dots\alpha_p}\varphi_{\alpha_1\dots\alpha_p}\,. \tag{3.55}$$

These can be used then to define the second order self-adjoint differential operators ("noncommutative laplacians")

$$\begin{aligned}
\Delta_1 &= -D\bar{D}\\
&= \tilde{\Gamma}\mathcal{D}A^{-1}\tilde{\mathcal{D}}\Gamma A\\
&= \tilde{\Gamma}\rho(d+\mathcal{B})\rho^{-1}A^{-1}\rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho\Gamma A\,, \qquad (3.56)\\
\Delta_2 &= -\bar{D}D\\
&= A^{-1}\tilde{\mathcal{D}}\Gamma A\tilde{\Gamma}\mathcal{D}\\
&= A^{-1}\rho^{-1}(\tilde{d}+\tilde{\mathcal{B}})\rho\Gamma A\tilde{\Gamma}\rho(d+\mathcal{B})\rho^{-1}\,. \qquad (3.57)
\end{aligned}$$

In the case $p=0$ these operators have the form

$$\begin{aligned}
\Delta_1 &= \tilde{\Gamma}\mathcal{D}\tilde{\mathcal{D}}\Gamma\\
&= \Gamma^\mu\rho(\partial_\mu+\mathcal{B}_\mu)\rho^{-2}(\partial_\nu+\mathcal{B}_\nu)\rho\Gamma^\nu\,, \qquad (3.58)\\
\Delta_2 &= \tilde{\mathcal{D}}\Gamma\tilde{\Gamma}\mathcal{D}\\
&= \rho^{-1}(\partial_\nu+\mathcal{B}_\nu)\rho\Gamma^\nu\Gamma^\mu\rho(\partial_\mu+\mathcal{B}_\mu)\rho^{-1}\,. \qquad (3.59)
\end{aligned}$$

In the elliptic case the above constructions can be used to develop noncommutative generalization of the standard Hodge-de Rham theory, in particular, noncommutative versions of the index theorems, the cohomology groups, the heat kernel etc. This is a very interesting topic that requires further study.

3.5 Differential operators for $p = 0$

In present paper we will restrict ourselves to the case $p = 0$, more precisely, to the second order self-adjoint differential operators acting on smooth sections of the bundle $\mathcal{V}[\frac{1}{2}]$ that can be presented in the form

$$L = -\rho^{-1}(\partial_\mu + \mathcal{B}_\mu)\rho a^{\mu\nu}\rho(\partial_\nu + \mathcal{B}_\nu)\rho^{-1} + Q\,, \tag{3.60}$$

by an appropriate choice of $a^{\mu\nu}$, ρ, $\mathcal{B}_\mu$ and $Q \in C^\infty(\mathrm{End}(\mathcal{V})[0])$. It is this operator that we will study in detail below.

This operator can be also written in the form

$$L = \bar{X}_\mu a^{\mu\nu} X_\nu + Q\,, \tag{3.61}$$

where

$$X_\mu = \partial_\mu + \mathcal{C}_\mu\,, \tag{3.62}$$

$$\bar{X}_\mu = -\partial_\mu + \bar{\mathcal{C}}_\mu\,, \tag{3.63}$$

and

$$\mathcal{C}_\mu = -\rho_{,\mu}\rho^{-1} + \rho\mathcal{B}_\mu\rho^{-1}\,, \tag{3.64}$$

$$\bar{\mathcal{C}}_\mu = -\rho^{-1}\rho_{,\mu} - \rho^{-1}\mathcal{B}_\mu\rho\,, \tag{3.65}$$

comma denotes partial derivatives, e.g. $\rho_{,\mu} = \partial_\mu\rho$,

In more details, it can be presented in the form

$$L = -a^{\mu\nu}\partial_\mu\partial_\nu + Y^\mu\partial_\mu + Z\,, \tag{3.66}$$

where

$$Y^\mu = -a^{\mu\nu}{}_{,\nu} + \bar{\mathcal{C}}_\nu a^{\mu\nu} - a^{\mu\nu}\mathcal{C}_\nu \tag{3.67}$$

$$Z = Q - (a^{\mu\nu}\mathcal{C}_\nu)_{,\mu} + \bar{\mathcal{C}}_\mu a^{\mu\nu}\mathcal{C}_\nu\,. \tag{3.68}$$

The leading symbol of the operator L is given by the matrix $H(x,\xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu$, with ξ a cotangent vector. In the section 2.3 we formulated precise conditions on this matrix for the operator L to be elliptic or hyperbolic. The system of hyperbolic partial differential equations describes the propagation of a *collection of waves* and generates the causal structure on the spacetime manifold [10, 11]. In the following we restrict ourselves for simplicity to the elliptic case, in which the leading symbol $H(x,\xi)$ is positive definite.

4. Examples

We see that in the matrix case the operator L does not define a unique riemannian metric $g^{\mu\nu}$. Rather there is a matrix-valued symmetric 2-tensor field $a^{\mu\nu}$. Although the matrix $g^{\mu\nu}$ plays the role of the riemannian metric in the commutative limit, it loses this role in fully noncommutative strongly deformed theory.

Abelian case. This is the case studied by Cutler and Wald [8, 9] and Boulanger et al. [15]. In this case the matrix $a^{\mu\nu}$ has the following form

$$a^{\mu\nu} = g^{\mu\nu}_{(a)}\Pi_{(a)}\,, \qquad \rho = g^{1/4}_{(a)}\Pi_{(a)}\,, \tag{4.1}$$

where $g_{(a)} = (\det g^{\mu\nu}_{(a)})^{-1}$, and $\Pi_{(a)}$ are the projections on the a-th component, that is diagonal matrices such that $(\Pi_{(a)})^A{}_B = \delta^A_B\delta^A_{(a)}\delta^{(a)}_B$ (no summation !). In this case all the structures commute and there is really nothing new in the matrix gravity theory, which is just the sum of the Einsten-Hilbert actions. What we consider in the present paper is *radically different* since it involves *fully noncommutative theory.* Some of the examples are considered below.

Commutative Limit. The simplest case is when there is a decomposition

$$a^{\mu\nu} = g^{\mu\nu}\mathbb{I} + \varkappa h^{\mu\nu}\,, \qquad \rho = g^{1/4}\exp(\varkappa\phi)\,, \tag{4.2}$$

where $\mathbb{I}$ is the unity matrix, $g^{\mu\nu}$ is a contravariant symmetric 2-tensor, $h^{\mu\nu}$ is a matrix-valued tensor field, ϕ is a matrix-valued scalar field and $\varkappa$ is a deformation parameter.

Reducible Case. A more general decomposition is given by

$$a^{\mu\nu} = g^{\mu\nu}\Omega + \varkappa h^{\mu\nu}\,, \qquad \rho = g^{1/4}\Omega^{-n/8}\exp(\varkappa\phi)\Omega^{-n/8}\,, \tag{4.3}$$

where Ω is a self-adjoint positive definite (non-constant) section of the endomorphism bundle $\mathrm{End}(\mathcal{V})[0]$, i.e. $\bar\Omega = \Omega$, $\Omega > 0$.

Commutative $N = 2$ Case. The simplest nontrivial model is when the matrix $a^{\mu\nu}$ is a real symmetric 2×2 matrix with the gauge group $O(2)$ and the fiber metric $E = \mathbb{I}$. We have then the following decomposition

$$a^{\mu\nu} = g^{\mu\nu}\mathbb{I} + \varkappa\tau h^{\mu\nu}\,, \qquad \rho = g^{1/4}\exp(\varkappa\tau\phi)\,, \tag{4.4}$$

where $h^{\mu\nu}$ is a tensor field, ϕ is a scalar and

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.5}$$

The condition of positivity of the matrix $H(x,\xi) = a^{\mu\nu}\xi_\mu\xi_\nu$ reads

$$\varkappa|h^{\mu\nu}\xi_\mu\xi_\nu| < |g^{\mu\nu}\xi_\mu\xi_\nu| \tag{4.6}$$

for any $\xi \neq 0$.

Noncommutative $N = 2$ Case. The simplest example of a fully deformed noncommutative theory is the case of $N = 2$ with the gauge group U(2). Assuming that the fiber metric is given by

$$E = \mathbb{I}\,, \tag{4.7}$$

we have the decomposition in terms of Pauli matrices

$$a^{\mu\nu} = g^{\mu\nu}\mathbb{I} + \varkappa\tau_a h_a^{\mu\nu}, \qquad \rho = g^{1/4}\exp(\varkappa\tau_a\phi_a), \tag{4.8}$$

where $h_a^{\mu\nu}$ are tensor fields, ϕ_a are scalars and τ_a are standard Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.9}$$

The condition of positivity of the matrix $H(x,\xi) = a^{\mu\nu}\xi_\mu\xi_\nu$ reads

$$\left(g^{\mu\nu}g^{\alpha\beta} - \varkappa^2 h_a^{\mu\nu}h_a^{\alpha\beta}\right)\xi_\mu\xi_\nu\xi_\alpha\xi_\beta > 0 \tag{4.10}$$

for any $\xi \neq 0$.

Large N Case. It is also interesting to consider the case of the U(N) gauge group in the limit $N \to \infty$. Since the matrix-valued metrics do not commute, it could serve as a toy model for quantum gravity. The decomposition has the form

$$a^{\mu\nu} = g^{\mu\nu}\mathbb{I} + \varkappa(N)h_a^{\mu\nu}T_a, \tag{4.11}$$

where T_a are the generators of SU(N) and one can adjust the dependence of the coupling $\varkappa$ on N to simplify the limit $N \to \infty$.

5. Spectral asymptotics

As we mentioned above we restrict ourselves to elliptic operators. All the geometric quantities we need for the gravity theory can be extracted from the spectrum of the elliptic operator L (3.60) (on a compact manifold M). The spectrum of the operator L is defined as usual by

$$L\varphi_k = \lambda_k\varphi_k, \qquad ||\varphi_k||_{L^2} = 1, \tag{5.1}$$

where $k = 1, 2, \ldots$. It is well known that a self-adjoint elliptic partial differential operator with positive definite leading symbol on a compact manifold without boundary has a discrete real spectrum bounded from below [26]. Since the operator L transforms covariantly under the diffeomorphisms as well as under the gauge transformations (2.10) the spectrum is *invariant* under these transformations.

To study the spectral asymptotics, i.e. the behavior of λ_k as $k \to \infty$, one introduces the spectral functions, in particular, the heat trace

$$\Theta(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k}, \tag{5.2}$$

and the zeta function

$$\zeta(s) = \mu^{2s}\sum_{k=1}^{\infty}(\lambda_k + m^2)^{-s}, \tag{5.3}$$

where each eigenvalue is taken with its multiplicity and μ is a renormalization parameter introduced to preserve dimensions. Here t is a real parameter, m^2 is a sufficiently large positive mass parameter (so that the operator $(L + m^2)$ is positive) and s is a complex parameter. It is well known that for a self-adjoint elliptic second-order partial differential operator with positive definite leading symbol the series (5.2) converges and defines a smooth function for $t > 0$ and the series (5.3) converges for $\mathrm{Re}\, s > n/2$ and defines a meromorphic function with simple isolated poles on the real line. These spectral functions are related by the Laplace-Mellin transform

$$\zeta(s) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-tm^2} \Theta(t)\,. \tag{5.4}$$

Moreover, it is also known that $\zeta(s)$ is analytic at $s = 0$, which enables one to define the regularized determinant of the operator $(F + m^2)$ (the one-loop effective action in quantum theory) [24]

$$\Gamma_{(1)} = \frac{1}{2} \log \mathrm{Det} \left(\frac{L + m^2}{\mu^2} \right) = -\frac{1}{2} \partial_s \zeta(s) \Big|_{s=0}\,. \tag{5.5}$$

It turns out that the study of the spectral asymptotics is equivalent to the study of the asymptotic expansion of the heat trace $\Theta(t)$ as $t \to 0$ and that of the zeta function and the effective action as $m^2 \to \infty$. It is well known that there is an asymptotic expansion of the heat trace invariant as $t \to 0^+$ [26]

$$\Theta(t) \sim (4\pi)^{-n/2} \sum_{k=0}^\infty t^{k-n/2} A_k\,, \tag{5.6}$$

and an asymptotic expansion of the zeta-function as $m \to \infty$

$$\zeta(s) \sim (4\pi)^{-n/2} \sum_{k=0}^\infty \frac{\Gamma(k + s - n/2)}{\Gamma(s)} \mu^{2s} m^{n-2s-2k} A_k\,, \tag{5.7}$$

where A_k are so-called heat invariants [26, 20, 16, 19, 18]. The coefficients A_k are spectral invariants determined by the integrals over the manifold of some local invariants [26, 16, 20] (for a review, see [19, 21]) constructed from the coefficients of the operator L and their derivatives so that they are polynomial in the derivatives of the coefficients of the operator L. The heat invariants A_k determine further the large mass asymptotic expansion of the effective action as $m \to \infty$ [16, 19, 20, 21]

$$\Gamma_{(1)} \sim c_0 m^n A_0 + c_1 m^{n-1} A_1 + \sum_{k=2}^\infty c_k m^{n-2k} A_k\,, \tag{5.8}$$

where

$$c_k = -\frac{1}{2} (4\pi)^{-n/2} \frac{\partial}{\partial s} \left[\frac{\Gamma(k + s - n/2)}{\Gamma(s)} \left(\frac{m}{\mu} \right)^{-2s} \right] \Bigg|_{s=0}\,. \tag{5.9}$$

One of the most powerful methods to study the spectral asymptotics is the heat kernel. The heat kernel, $U(t|x, x')$, is defined as the kernel of the heat semigroup $\exp(-tL)$ operator for $t > 0$, i.e. the fundamental solution of the heat equation

$$(\partial_t + L) U(t|x, x') = 0 \tag{5.10}$$

with the initial condition

$$U(0^+|x,x') = \delta(x-x')\,, \tag{5.11}$$

$\delta(x-x')$ being the Dirac distribution.

For $t>0$ the heat kernel $U(t|x,x')$ is a smooth function near the diagonal of $M\times M$ and has a well defined diagonal value

$$U(t|x,x) = \sum_{k=1}^{\infty} e^{-t\lambda_k}\varphi_k(x)\otimes\bar\varphi_k(x)\,, \tag{5.12}$$

which has the asymptotic expansion as $t\to 0$ [26]

$$U(t|x,x) \sim (4\pi)^{-n/2}\sum_{k=0}^{\infty} t^{k-n/2}a_k(x)\,, \tag{5.13}$$

where $a_k(x)$ are some covariant densities.

Moreover, the heat semigroup $\exp(-tL)$ is a trace-class operator with a well defined L^2 trace

$$\mathrm{Tr}_{L^2}\exp(-tL) = \int\limits_M dx\;\mathrm{tr}_V\,U(t|x,x) = \Theta(t)\,, \tag{5.14}$$

where tr_V denotes the fiber trace. We have defined the heat kernel in such a way that it transforms as a density of weight $\frac{1}{2}$ at both points x and x'. More precisely, it is a section of the exterior tensor product bundle $\mathcal{V}[\frac{1}{2}]\boxtimes\mathcal{V}^*[\frac{1}{2}]$. Therefore, the heat kernel diagonal transforms as a density of weight 1, i.e. it is a section of the bundle $\mathrm{End}(\mathcal{V})[1]$, and the trace $\mathrm{Tr}_{L^2}\exp(-tL)$ is invariant under diffeomorphisms.

Therefore, the heat invariants A_k can be constructed by computing the $t\to 0$ asymptotics of the solution of the heat equation and integrating the coefficients $a_k(x)$ over the manifold, i.e.

$$A_k = \int\limits_M dx\;\mathrm{tr}_V\,a_k(x)\,. \tag{5.15}$$

A second-order differential operator is called Laplace type if it has a scalar leading symbol. Most of the calculations in quantum field theory and spectral geometry are restricted to the Laplace type operators for which nice theory of heat kernel asymptotics is available [26, 16, 18, 19, 20, 21]. However, the operators condidered in the present paper have a matrix valued principal symbol $H(x,\xi)$ and are, therefore, not of Laplace type. The study of heat kernel asymptotics for non-Laplace type operators is quite new and the methodology is still underdeveloped. As a result even the first heat invariants (A_0, A_1 and A_2) are not known in general. For some partial results see [27, 22, 23].

5.1 Calculation of the invariants A_0 and A_1

For so called natural non-Laplace type differential operators, which are constructed from a riemannian metric and canonical connections on spin-tensor bundles the coefficients A_0 and A_1 were computed in [23]. Following this paper we will use a *formal* method that is sufficient for our purposes of computing the asymptotics of the heat trace of the second-order elliptic self-adjoint operator L (3.60).

First, we present the heat kernel diagonal in the form

$$U(t|x,x) = \int\limits_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} e^{-i\xi x} \exp(-tL) e^{i\xi x}\,, \tag{5.16}$$

where $\xi x = \xi_\mu x^\mu$, which can be transformed to

$$U(t|x,x) = \int\limits_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \exp\left[-t\,(H+K+L)\right]\cdot\mathbb{I}\,, \tag{5.17}$$

where H is the leading symbol of the operator L

$$H = a^{\mu\nu}\xi_\mu\xi_\nu\,, \tag{5.18}$$

and K is a first-order self-adjoint operator defined by

$$K = i\xi_\mu\left(\bar{X}_\nu a^{\mu\nu} - a^{\mu\nu}X_\nu\right), \tag{5.19}$$

where the operator X_μ is defined by (3.62). Here the operators in the exponent act on the unity matrix $\mathbb{I}$ from the left. By changing the integration variable $\xi \to t^{-1/2}\xi$ we obtain

$$U(t|x,x) = (4\pi t)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \exp\left(-H - \sqrt{t}K - tL\right)\cdot\mathbb{I}\,, \tag{5.20}$$

and the problem becomes now to evaluate the first three terms of the asymptotic expansion of this integral as $t \to 0$. By using the Volterra series

$$\begin{aligned}\exp(A+B) &= e^A + \sum_{k=1}^{\infty} \int\limits_0^1 d\tau_k \int\limits_0^{\tau_k} d\tau_{k-1} \cdots \int\limits_0^{\tau_2} d\tau_1 \times \\ &\quad\times e^{(1-\tau_k)A} B e^{(\tau_k-\tau_{k-1})A} \cdots e^{(\tau_2-\tau_1)A} B e^{\tau_1 A}\,,\end{aligned} \tag{5.21}$$

we get

$$\begin{aligned}\exp\left(-H - \sqrt{t}K - tL\right) &= e^{-H} - t^{1/2}\int\limits_0^1 d\tau_1 e^{-(1-\tau_1)H} K e^{-\tau_1 H} + \\ &\quad + t\left[\int\limits_0^1 d\tau_2 \int\limits_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \right. \\ &\quad \left. - \int\limits_0^1 d\tau_1 e^{-(1-\tau_1)H} L e^{-\tau_1 H}\right] + O(t^2)\,.\end{aligned} \tag{5.22}$$

Now, since K is linear in ξ the term proportional to $t^{1/2}$ vanishes after integration over ξ. Thus, we obtain the first two coefficients of the heat kernel diagonal

$$U(t|x,x) \sim (4\pi t)^{-n/2}\left[a_0 + t a_1 + O(t^2)\right] \tag{5.23}$$

in the form

$$a_0 = \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} e^{-H}, \tag{5.24}$$

$$a_1 = \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \left[\int\limits_0^1 d\tau_2 \int\limits_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \right.$$
$$\left. - \int\limits_0^1 d\tau_1 e^{-(1-\tau_1)H} L e^{-\tau_1 H} \right]. \tag{5.25}$$

These quantities are matrix densities that define finally the heat invariants

$$A_0 = \int\limits_M dx \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \operatorname{tr}_V e^{-H}, \tag{5.26}$$

$$A_1 = \int\limits_M dx \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \operatorname{tr}_V \left[\int\limits_0^1 d\tau_2 \int\limits_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)H} K e^{-(\tau_2-\tau_1)H} K e^{-\tau_1 H} - \right.$$
$$\left. - \int\limits_0^1 d\tau_1 e^{-(1-\tau_1)H} L e^{-\tau_1 H} \right]. \tag{5.27}$$

These quantities are invariant under both the diffeomorphisms and the gauge transformations. The coefficient a_0 is constructed from the matrix a but not its derivatives, whereas the coefficient a_1 is constructed from the matrix a and its first and second derivatives as well as from the first derivatives of the field $\mathcal{B}$ and the matrix ρ and its first and second derivatives; obviously, it is linear in the matrix Q. Morevover, it does not depend on the derivatives of Q and is polynomial in the derivatives of $a^{\mu\nu}$, ρ and $\mathcal{B}_\mu$, more precisely, linear in second derivatives of a and ρ and the first derivatives of $\mathcal{B}$ and quadratic in first derivatives of a and ρ. Further, since the operator L is self-adjoint, the heat kernel diagonal is self-adjoint matrix density and the heat trace is a real invariant. Therefore, the coefficients a_0 and a_1 are self-adjoint, and the invariants A_0 and A_1 are real.

By integrating by parts and using the cyclic property of the trace one can simplify the expression for the invariant A_1 as follows

$$A_1 = \int\limits_M dx \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \operatorname{tr}_V \left[-e^{-H} Q - \int\limits_0^1 d\tau_1 \overline{W_\mu(1-\tau_1)} a^{\mu\nu} W_\nu(\tau_1) \right.$$
$$\left. + \int\limits_0^1 d\tau_2 \int\limits_0^{\tau_2} d\tau_1 \overline{T(1-\tau_2)} e^{-(\tau_2-\tau_1)H} T(\tau_1) \right], \tag{5.28}$$

where

$$W_\mu(\tau) = X_\mu e^{-\tau H}, \tag{5.29}$$
$$T(\tau) = K e^{-\tau H}. \tag{5.30}$$

Finally, by using the Duhammel formula

$$\partial e^{A} = \int_0^1 ds\, e^{(1-s)A}(\partial A)e^{sA} \tag{5.31}$$

we can compute the derivatives of the exponential $e^{-\tau H}$

$$\partial_\mu e^{-\tau H} = -\int_0^\tau ds\, e^{-(\tau-s)H} H_\mu e^{-sH}\,, \tag{5.32}$$

where

$$H_\mu = a^{\alpha\beta}{}_{,\mu}\xi_\alpha\xi_\beta\,. \tag{5.33}$$

We can use this formula to compute the second derivatives of $e^{-\tau H}$ needed for the local coefficient a_1

$$\begin{aligned}\partial_\mu\partial_\nu e^{-\tau H} &= -\int_0^\tau ds_1\, e^{-(\tau-s_1)H} H_{\mu\nu} e^{-s_1 H} + \\ &+ \int_0^\tau ds_2 \int_0^{s_2} ds_1 \Big[e^{-(s_2-s_1)H} H_\nu e^{-s_1 H} H_\mu e^{-(\tau-s_2)H} + \\ &\qquad + e^{-(\tau-s_2)H} H_\mu e^{-(s_2-s_1)H} H_\nu e^{-s_1 H} \Big]\,,\end{aligned} \tag{5.34}$$

where

$$H_{\mu\nu} = a^{\alpha\beta}{}_{,\mu\nu}\xi_\alpha\xi_\beta\,. \tag{5.35}$$

By isolating the overall exponential factor we get

$$\partial_\mu e^{-\tau H} = -\beta_\mu(\tau) e^{-\tau H}\,, \tag{5.36}$$

where

$$\beta_\mu(\tau) = \int_0^\tau ds\, e^{-sH} H_\mu e^{sH}\,, \tag{5.37}$$

which can be presented in the algebraic form

$$\beta_\mu(\tau) = \frac{e^{-\tau Ad_H}-1}{Ad_H} H_\mu = \sum_{k=1}^{\infty} \frac{(-\tau)^k}{k!} \underbrace{[H, \cdots [H}_{k-1}, H_\mu]\cdots]\,. \tag{5.38}$$

Here Ad_H is the operator defined by

$$Ad_H A = [H, A]\,. \tag{5.39}$$

By using the above formulas we obtain

$$W_\mu(\tau) = [\mathcal{C}_\mu - \beta_\mu(\tau)]e^{-\tau H}\,, \tag{5.40}$$

$$T(\tau) = i[\alpha + 2J^\nu \beta_\nu(\tau)]e^{-\tau H}\,, \tag{5.41}$$

where

$$J^\nu = a^{\mu\nu}\xi_\mu\,, \tag{5.42}$$

$$\alpha = \bar{\mathcal{C}}_\nu J^\nu - J^\nu \mathcal{C}_\nu - \xi_\mu a^{\mu\nu}{}_{,\nu}\,. \tag{5.43}$$

Thus, finally, summarizing this section we formulate the results in form of a theorem.

Theorem 1. *The heat trace of the operator L (3.60) has the asymptotic expansion as $t \to 0^+$*

$$\mathrm{Tr}_{L^2} \exp(-tL) = (4\pi t)^{-n/2}\left[A_0 + tA_1 + O(t^2)\right], \tag{5.44}$$

where

$$A_0 = \int\limits_M dx \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}}\, \mathrm{tr}_V\, e^{-H}\,, \tag{5.45}$$

$$\begin{aligned} A_1 = \int\limits_M dx \int\limits_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}}\, \mathrm{tr}_V\, e^{-H} \Bigg\{ &-Q - \int\limits_0^1 d\tau_1 [\bar{\mathcal{C}}_\mu - \bar{\beta}_\mu(1-\tau_1)] a^{\mu\nu} [\mathcal{C}_\nu - \beta_\nu(\tau_1)] + \\ &+ \int\limits_0^1 d\tau_2 \int\limits_0^{\tau_2} d\tau_1 e^{(\tau_2-\tau_1)H} [\bar{\alpha} + 2\bar{\beta}_\nu(1-\tau_2)J^\nu] e^{-(\tau_2-\tau_1)H} \times \\ &\times [\alpha + 2J^\mu \beta_\mu(\tau_1)] \Bigg\}\,. \end{aligned} \tag{5.46}$$

Here $\mathcal{C}_\mu$, β_μ, α and J^ν are given by eqs. (3.64), (5.33), (5.37), (5.42) and (5.43).

5.2 Commutative limit

Let us compute the coefficient A_1 in the commutative limit. We let

$$a^{\mu\nu} = g^{\mu\nu}\mathbb{I}\,, \qquad \rho = g^{1/4}e^{\phi}\mathbb{I}\,, \qquad Q = q\mathbb{I}\,, \tag{5.47}$$

where $g^{\mu\nu}$ is a nonsingular matrix, $g = (\det g^{\mu\nu})^{-1}$, and ϕ is a scalar function. The operator L (3.60) has the form

$$\begin{aligned} L &= -g^{-1/4}(\partial_\mu + \mathcal{B}_\mu + \phi_{,\mu}) g^{1/2} g^{\mu\nu} (\partial_\nu + \mathcal{B}_\nu - \phi_{,\nu}) g^{-1/4} + q \\ &= -g^{-1/4}(\partial_\mu + \mathcal{B}_\mu) g^{1/2} g^{\mu\nu} (\partial_\nu + \mathcal{B}_\nu) g^{-1/4} + \tilde{q}\,, \end{aligned} \tag{5.48}$$

where

$$\tilde{q} = q + \Box\phi + g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\,, \tag{5.49}$$

and

$$\Box = g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu\,. \tag{5.50}$$

Then

$$H = g^{\mu\nu}\xi_\mu\xi_\nu\,, \qquad H_\mu = g^{\alpha\beta}{}_{,\mu}\xi_\alpha\xi_\beta\,, \qquad J^\mu = g^{\mu\nu}\xi_\nu\,, \tag{5.51}$$

$$\mathcal{C}_\mu = -\phi_{,\mu} - \frac{1}{4}(\log g)_{,\mu} + \mathcal{B}_\mu\,, \qquad \bar{\mathcal{C}}_\mu = -\phi_{,\mu} - \frac{1}{4}(\log g)_{,\mu} - \mathcal{B}_\mu\,, \tag{5.52}$$

$$\alpha = -\xi_\mu g^{\mu\nu}{}_{,\nu} - 2\xi_\mu g^{\mu\nu}\mathcal{B}_\nu\,, \qquad \bar{\alpha} = -\xi_\mu g^{\mu\nu}{}_{,\nu} + 2\xi_\mu g^{\mu\nu}\mathcal{B}_\nu\,, \tag{5.53}$$

$$\beta_\mu(\tau) = \tau H_\mu = \tau g^{\alpha\beta}{}_{,\mu}\xi_\alpha\xi_\beta\,. \tag{5.54}$$

The integrals over the cotangent vector ξ are simply gaussian and can be easily computed. First of all, we have

$$\int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}}\, e^{-H} = g^{1/2}\,. \tag{5.55}$$

So, we immediately obtain the coefficient A_0 as the riemannian volume of the manifold

$$A_0 = \int_M dx\, g^{1/2}\,. \tag{5.56}$$

Next, we introduce the notation for the gaussian averages

$$\langle f \rangle = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}}\, g^{-1/2} e^{-H} f(\xi)\,. \tag{5.57}$$

Then the gaussian average of an exponential function gives the generating function

$$\langle \exp(\xi x) \rangle = \exp\left(\frac{1}{4} g_{\mu\nu} x^\mu x^\nu\right), \tag{5.58}$$

where $g_{\mu\nu}$ is the inverse matrix of the matrix $g^{\mu\nu}$. Expansion in the power series in x generates the gaussian averages of polynomials

$$\left\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n+1}} \right\rangle = 0, \tag{5.59}$$

$$\left\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n}} \right\rangle = \frac{(2n)!}{2^{2n} n!} g_{(\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n})}\,, \tag{5.60}$$

where the parenthesis denote complete symmetrization over all indices. In particular,

$$\langle 1 \rangle = 1\,, \qquad \langle \xi_\mu \xi_\nu \rangle = \frac{1}{2} g_{\mu\nu}\,, \tag{5.61}$$

$$\begin{aligned} \langle \xi_\mu \xi_\nu \xi_\alpha \xi_\beta \rangle &= \frac{3}{4} g_{(\mu\nu} g_{\alpha\beta)} \\ &= \frac{1}{4} (g_{\mu\nu} g_{\alpha\beta} + g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha})\,, \end{aligned} \tag{5.62}$$

$$\begin{aligned} \langle \xi_\mu \xi_\nu \xi_\alpha \xi_\beta \xi_\rho \xi_\sigma \rangle &= \frac{15}{8} g_{(\mu\nu} g_{\alpha\beta} g_{\rho\sigma)} \\ &= \frac{1}{8} (g_{\mu\nu} g_{\alpha\beta} g_{\rho\sigma} + g_{\mu\alpha} g_{\nu\beta} g_{\rho\sigma} + g_{\mu\beta} g_{\nu\alpha} g_{\rho\sigma} + g_{\mu\rho} g_{\alpha\beta} g_{\nu\sigma} + g_{\mu\sigma} g_{\alpha\beta} g_{\rho\nu} + \\ &\quad + g_{\mu\nu} g_{\alpha\rho} g_{\beta\sigma} + g_{\mu\alpha} g_{\nu\rho} g_{\beta\sigma} + g_{\mu\beta} g_{\nu\rho} g_{\alpha\sigma} + g_{\mu\rho} g_{\alpha\nu} g_{\beta\sigma} + g_{\mu\sigma} g_{\alpha\rho} g_{\beta\nu} + \\ &\quad + g_{\mu\nu} g_{\alpha\sigma} g_{\beta\rho} + g_{\mu\alpha} g_{\nu\sigma} g_{\beta\rho} + g_{\mu\beta} g_{\nu\sigma} g_{\alpha\rho} + g_{\mu\rho} g_{\alpha\sigma} g_{\beta\nu} + g_{\mu\sigma} g_{\alpha\nu} g_{\beta\rho})\,. \end{aligned} \tag{5.63}$$

We have

$$\begin{aligned} A_1 = \int_M dx\, g^{1/2}\, \mathrm{tr}_V \Bigg\langle -q - \int_0^1 d\tau_1 [\bar{\mathcal{C}}_\mu - (1-\tau_1) H_\mu] g^{\mu\nu} (\mathcal{C}_\nu - \tau_1 H_\nu) \\ + \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 [\bar{\alpha} + 2(1-\tau_2) h](\alpha + 2\tau_1 h) \Bigg\rangle\,, \end{aligned} \tag{5.64}$$

where

$$h = J^\mu H_\mu = g^{\mu\rho} g^{\alpha\beta}{}_{,\mu} \xi_\alpha \xi_\beta \xi_\rho \,. \tag{5.65}$$

Evaluating the integrals over the parameters τ_1 and τ_2 we obtain

$$\begin{aligned} A_1 = \int\limits_M dx\, g^{1/2} \, \mathrm{tr}_V \Bigg\langle & -q + (g^{\mu\nu} - 2J^\mu J^\nu)\mathcal{B}_\mu \mathcal{B}_\nu - \\ & - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - g^{\mu\nu}\left[H_\mu + \frac{1}{2}(\log g)_{,\mu}\right]\phi_{,\nu} - \\ & - \frac{1}{4} g^{\mu\nu} H_\mu (\log g)_{,\nu} - \frac{1}{16} g^{\mu\nu} (\log g)_{,\mu} (\log g)_{,\nu} - \\ & - \frac{1}{6} g^{\mu\nu} H_\mu H_\nu + \frac{1}{2}\lambda^2 - \frac{2}{3} h\lambda + \frac{1}{6} h^2 \Bigg\rangle \,, \end{aligned} \tag{5.66}$$

where

$$\lambda = \xi_\mu g^{\mu\nu}{}_{,\nu} \,. \tag{5.67}$$

Lastly, we need to evaluate the gaussian averages. By taking into account the formulas above we obtain

$$\langle J^\mu J^\nu \rangle = \frac{1}{2} g^{\mu\nu} \,, \tag{5.68}$$

$$\langle H_\mu \rangle = \frac{1}{2} g_{\alpha\beta} g^{\alpha\beta}{}_{,\mu} = -\frac{1}{2} (\log g)_{,\mu} \,, \tag{5.69}$$

$$\langle H_\mu H_\nu \rangle = \frac{1}{4} (\log g)_{,\mu} (\log g)_{,\nu} + \frac{1}{2} g_{\rho\alpha} g^{\alpha\beta}{}_{,\mu} g_{\beta\sigma} g^{\sigma\rho}{}_{,\nu} \,, \tag{5.70}$$

$$\langle \lambda^2 \rangle = \frac{1}{2} g_{\mu\nu} g^{\mu\alpha}{}_{,\alpha} g^{\nu\beta}{}_{,\beta} \,, \tag{5.71}$$

$$\langle h\lambda \rangle = -\frac{1}{4} (\log g)_{,\mu} g^{\mu\alpha}{}_{,\alpha} + \frac{1}{2} g_{\mu\nu} g^{\mu\alpha}{}_{,\alpha} g^{\nu\beta}{}_{,\beta} \,, \tag{5.72}$$

$$\begin{aligned} \langle h^2 \rangle &= \frac{15}{8} g_{(\mu\nu} g_{\alpha\beta} g_{\rho\sigma)} g^{\gamma\rho} g^{\delta\sigma} g^{\mu\nu}{}_{,\gamma} g^{\alpha\beta}{}_{,\delta} \\ &= \frac{1}{8}\Big[g^{\gamma\delta} (\log g)_{,\gamma} (\log g)_{,\delta} - 4 g^{\mu\nu}{}_{,\mu} (\log g)_{,\nu} + \\ &\qquad + 2 g_{\mu\alpha} g_{\nu\beta} g^{\gamma\delta} g^{\mu\nu}{}_{,\gamma} g^{\alpha\beta}{}_{,\delta} + 4 g_{\mu\alpha} g^{\mu\nu}{}_{,\nu} g^{\alpha\beta}{}_{,\beta} + 4 g_{\beta\nu} g^{\mu\nu}{}_{,\alpha} g^{\alpha\beta}{}_{,\mu} \Big] \,. \end{aligned} \tag{5.73}$$

Here we used the equation

$$(\log g)_{,\mu} = -g_{\alpha\beta} g^{\alpha\beta}{}_{,\mu} = g^{\alpha\beta} g_{\alpha\beta,\mu} \,. \tag{5.74}$$

We see that A_1 does not depend on $\mathcal{B}$ at all and we get finally

$$\begin{aligned} A_1 = \int\limits_M dx\, g^{1/2} N \Bigg\{ & -q - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \\ & + \frac{1}{24} g^{\mu\nu} (\log g)_{,\mu} (\log g)_{,\nu} + \frac{1}{12} (\log g)_{,\mu} g^{\mu\nu}{}_{,\nu} + \\ & + \frac{1}{12} g_{\beta\nu} g^{\mu\nu}{}_{,\alpha} g^{\alpha\beta}{}_{,\mu} - \frac{1}{24} g_{\mu\alpha} g_{\nu\beta} g^{\gamma\delta} g^{\mu\nu}{}_{,\gamma} g^{\alpha\beta}{}_{,\delta} \Bigg\} \,. \end{aligned} \tag{5.75}$$

One can further integrate by parts here but we will stop here. Since the part that depends only on g must be an invariant, it must be an integral of a scalar quantity, which can be only the scalar curvature (there is only one such scalar). It turns out that

$$\begin{aligned} A_1 &= \int\limits_M dx\, g^{1/2} N \left(-q - g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \frac{1}{6} R \right) \\ &= \int\limits_M dx\, g^{1/2} N \left(-\tilde{q} + \frac{1}{6} R \right), \end{aligned} \tag{5.76}$$

where R is the scalar curvature of the metric g and $\tilde{q}$ is defined by (5.49). This coincides with the standard results [26, 19, 21] for the Laplace operator.

6. Noncommutative gauged gravity

Our main idea is that gravity has some *new degrees of freedom*. Instead of describing gravity by one tensor field we propose to describe it by new dynamical variables: the matrix valued tensor field $a^{\mu\nu}$ and the density matrix field ρ.

6.1 Matter in gravitational field

The motion of a massive particle in the gravitational field should be determined by the principle of the extremal action. In general relativity the action is simply the interval $S = -m \int \sqrt{-ds^2}$, which means that particles propagate along the geodesics of the metric $g_{\mu\nu}$. The parameter m is the mass of the particle. In our modified theory, we assume that a particle "splits" into several parts which propagate separately and have different masses. In other words, we propose to describe the motion of a particle in gravitational field by the action

$$S_{\text{particle}} = -\sum_{i=1}^{s} m_{(i)} \int \sqrt{-ds^2_{(i)}}\,, \tag{6.1}$$

where the intervals $ds^2_{(i)}$ are defined by (2.37). In the commutative limit the sum of the mass parameters determines the usual mass

$$m = \sum_{i=1}^{s} m_{(i)}\,. \tag{6.2}$$

One can develop the whole Hamilton-Jacobi formalism starting from this for the particle mechanics based on Finsler geometry. We will not do it here but refer the reader to Rund's book [28] (in particular, chapter 7 of the russian edition).

The dynamics of matter fields in gravitational field can be now described by the differential calculus developed in section 3. A typical action for the matter fields has the form

$$S_{\text{field}} = \int\limits_M dx \left\{ -\langle \varphi, L\varphi \rangle + W(\varphi) \right\}, \tag{6.3}$$

where L is an appropriate second-order differential operator constructed by the methods of section 3 and W is a potential. The first-order operators of Dirac type needed for spinor fields can be constructed similarly.

6.2 Equations of gravitational field

Our goal is to construct an action functional that is invariant under both the diffeomorphisms and the local gauge transformations. We also require that it is a noncommutative deformation of the Einstein general relativity, which means that our action should reduce to the standard Einstein-Hilbert functional in the commutative limit.

We construct the classical action following the ideology of "induced gravity" as the large mass limit of the effective action. The classical action of gravity is then identified with the first two coefficients of the asymptotic expansion of the effective action as $m \to \infty$ (5.8). In other words, we use the coefficients A_0 and A_1 to construct the invariant action functional of the noncommutative gravity. For a dynamical theory we need an invariant action functional that depends on the first derivatives of dynamical degrees of freedom. Such an invariant is given by the heat kernel coefficient A_1. Therefore, the invariant action functional of noncommutative gravity is a linear combination of the first two heat kernel coefficients

$$S = \frac{1}{16\pi G}\frac{1}{N}\left[6\, A_1 - 2\Lambda\, A_0\right] + S_{\text{matter}}\,, \tag{6.4}$$

where S_{matter} is the action of matter fields, G and Λ are the phenomenological coupling constants (Newton constant and the cosmological constant), and the coefficients are chosen in such a way that it has the correct commutative limit. This action is in some sort unique and the dynamical model described by it can be called the *Induced Noncommutative Gauged Gravity*.

It is worth indicating the general form of the action. It can be represented symbolically as follows

$$\begin{aligned} S = \frac{1}{16\pi G N}\int\limits_M dx\,\mathrm{tr}_V \Big\{ & F(a,\rho)\partial a\partial a + F(a,\rho)\partial a\partial\rho + F(a,\rho)\partial\rho\partial\rho + \\ & + F(a)(6Q + 2\Lambda) + F(a,\rho)\partial\mathcal{B} + F(a,\rho)\mathcal{B}\mathcal{B}\Big\} + S_{\text{matter}}\,, \end{aligned} \tag{6.5}$$

where ∂ denotes the derivatives and $F(a,\rho)$ denotes the coefficients that can only depend on a and ρ. Of course, all the coefficients are different and one has to include similar terms with all possible orderings of noncommutative factors.

The fundamental equations of this model are

$$\frac{\delta S}{\delta a^{\mu\nu}} = 0\,, \qquad \frac{\delta S}{\delta\rho} = 0\,, \qquad \frac{\delta S}{\delta\mathcal{B}_\mu} = 0\,. \tag{6.6}$$

Notice that the gravitational action does not depend on the derivatives of the vector field $\mathcal{B}_\mu$ (after integration by parts). Therefore, if the matter action S_{matter} does not include a kinetic term for the fields $\mathcal{B}$, then the variation with respect to $\mathcal{B}$ gives just a constraint, which simply expresses $\mathcal{B}$ in terms of the derivatives of the matrices $a^{\mu\nu}$ and ρ. Also, the matrix Q is fixed as a given self-adjoint positive definite function of a and ρ and their derivatives. Its form can be adjusted by imposing some additional physical requirements. This could be important for spontaneous symmetry breakdown. For simplicity, it can be just set to zero $Q = 0$.

If we introduce a deformation parameter $\varkappa$ according to (5.47): $a^{\mu\nu} = g^{\mu\nu}\mathbb{I} + \varkappa h^{\mu\nu}$, $\rho = g^{1/4}\exp(\mathbb{I}\phi + \varkappa\psi)$, and $Q = q\mathbb{I} + \varkappa P$, where $h^{\mu\nu}$ is a matrix-valued tensor field, ϕ and q are scalar fields, and ψ and P are matrix-valued scalar fields, then in the limit $\varkappa \to 0$ the action becomes

$$S_{(0)} = \frac{1}{16\pi G}\int\limits_M dx\, g^{1/2}\left(R - 2\Lambda - 6g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 6q\right) + S_{\mathrm{matter}(0)}\,, \tag{6.7}$$

which describes the Einstein general relativity and a scalar field. Note that in the case $q = -g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \Box\phi$ the scalar field disappears. Other interesting choices of q are $q = m^2$ or $q = \Lambda(\phi^2 - m^2)^2$.

It would be certainly very interesting to compute the deformation corrections to the action (6.7), that is to understand the precise way under which the commutative action (6.7) gets deformed to the non-commutative one (6.4). In general, we should obtain

$$S(a,\rho,\mathcal{B}) = S_{(0)}(g,\phi) + \sum_{k=1}^{\infty}\varkappa^k S_{(k)}(g,\phi,h,\psi,\mathcal{B})\,. \tag{6.8}$$

Notice that the zero-order action $S_{(0)}$ does not depend on the fields $\mathcal{B}$. By a simple rescaling $\phi \to \varkappa\phi$ and $q \to \varkappa q$ we could easily shift the dependence on these fields to the higher orders as well so that the zero order term $S_{(0)}(g)$ is just the Einstein-Hilbert action. The coefficients $S_{(k)}$ are polynomial in the fields $\phi, h, \mathcal{B}$ and ψ. Their general form can be read off from (6.5). In principle, one could get the coefficients $S_{(k)}$ from the general result of Theorem 1. However, such a calculation (even if straightforward) presents a real technical challenge and would require a separate paper. We plan to carry out this calculation in the near future and present the results elsewhere.

7. Discussion

In conclusion we list some interesting open problems in the proposed model.

Uniqueness of the noncommutative deformation. First of all, it is very important to understand whether the proposed deformation of the general relativity is unique. If it is not unique, then what additional physical conditions should one impose to make such a deformation unique.

Interaction of noncommutative gravity with matter. One needs to find a consistent way to describe the interaction of the ordinary matter with gravity. In particular, to understand which physical matter fields should interact with the noncommutative (gravicolor) degrees of freedom and which should only feel the graviwhite part.

Classical solutions and singularities. It would be very interesting problem to study simple solutions of this model, say a static spherically symmetric solution, which would describe a “noncommutative black hole” as well as a time-dependent homogeneous solution, which would describe the “noncommutative cosmology”. One should study the problem of singularities of classical solutions. If this model is free from singularities, it would be a very significant argument in favor of it.

Spontaneous breakdown of symmetry. One needs to understand whether it is possible to introduce the spontaneous breakdown of the gauge symmetry, so that in the broken phase in the vacuum there is just one tensor field, which is identified with the metric of the space-time. All other tensor fields must have zero vacuum expectation values. In the unbroken phase there will not be a metric at all in the usual sense since there is no preferred tensor field with non-zero vacuum expectation value.

Quantization and renormalization. Our model is, in fact, nothing but a generalized sigma model. So, the problems in quantization of this model are the same as in the quantization of the sigma model.

Semi-classical (one-loop) approximation and heat kernel asymptotics. We point out that the study of the one-loop approximation requires new calculational methods since the partial differential operators involved are not of the so-called Laplace type (nonscalar leading symbol). For example, even the heat kernel coefficients A_2 needed for the renormalization in four dimensions is not known in general.

High-energy behavior. We expect that the behavior of our model at higher energies should be radically different from the Einstein gravity since there is no preferred metric in the unbroken phase, when the new gauge symmetry is intact.

Dark energy. It would be very interesting to study the question whether the new noncommutative degrees of freedom of gravity could be accounted for the dark energy in cosmology.

Low energy behavior and confinement. One could expect the gauge (*gravicolor*) degrees of freedom to be confined within some short characteristic scales (say, Planck scale), so that only the invariants (*graviwhite* states) are visible at large distances. Then the metric and the curvature would be only effective characterictics of the spacetime at large distances.

Noncommutative deformation of riemannian geometry. The Einstein spaces are manifolds with the metric satisfying the vacuum Einstein equations (with cosmological constant), i.e. $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with some constant Λ. In other words, the Einstein metrics are the extremals of the Einstein-Hilbert functional. The study of Einstein spaces is a very important subject in riemannian geometry. It would be very interesting to study the noncommutative deformations of Einstein spaces defined as the extremals of the functional of matrix gravity. This would also have deep connections to "*noncommutative spectral geometry*".

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